Acknowledgments

Part of the work reported in this paper was performed with the support of the Department of the Air Force at Lincoln Laboratory. The views and conclusions contained in this document should not be interpreted as necessarily representing the official policies, either expressed or implied, of the U.S. Government.

References

[1] Vigneron, F.R. and Millar, R.A., "Flight Performance of the Stabilization System of the Communications Technology Satellite," *Journal of Guidance and Control*, Vol. 1, Nov.-Dec. 1978, pp. 404-412.

[2] Weber, R., "The Passive, Ground-Based, Electro-Optical Detection of Synchronous Satellites," TN-1978-27, ESD-TR-78-213, MIT Lincoln Laboratory, Lexington, Mass., June 1978.

[3] Vigneron, F.R. and Krag, W.E., "The Attitude Dynamics of Hermes After Termination of Three-Axis Stabilization," CRC Report No. 1354, Dept. of Communications, Ottawa, Canada, 1982.

AIAA 82-4257

Discrete-Time Stability of Continuous-Time Controller Designs for Large Space Structures

Mark J. Balas*
Rensselaer Polytechnic Institute, Troy, N.Y.

I. Introduction

IN most of the stable control designs for flexible structures, continuous time is assumed. Yet, it is well known that the controller will be implemented by one or more on-line digital computers; hence the discrete-time stability of such controllers is an important consideration.

For example, it is known the direct-velocity feedback (DVFB), i.e., negative feedback from collocated force actuators and velocity sensors, is a purely dissipative control law and hence cannot produce residual mode instabilities when it is designed for a reduced-order model of a structure.[1-3] However, the stability results are only known for continuous time; it is not immediately clear how much delay due to digital implementation of DVFB can be tolerated without loss of stability. In this Note, we will present some new results which answer this question in the sense that bounds on the time-step size will be given.

The class of distributed parameter systems, which includes flexible structures, may be described by the following:

$$v_t = Av + Bf \qquad v(0) = v_0 \qquad y = Cv \tag{1}$$

where, for each positive time t, the (possibly vector-valued) system state $v(t)$ is in H, an appropriate Hilbert space with inner product $(.,.)$ and corresponding norm $\|.\|$. The operator A is a time-invariant differential operator whose domain $D(A)$ is dense in H. A generates a C_0 semigroup $U(t)$ on H which is *dissipative* in the following sense:

$$\|U(t)\| \le M_0 e^{-\sigma_0 t} \qquad (t \ge 0) \tag{2}$$

where $M_0 \ge 1$ and $\sigma_0 > 0$ represents the total natural dissipation in the system. The input operator B has rank M (i.e., M inputs) and the output operator C has rank P (i.e., P outputs); these operators are determined by the type and location of actuators and sensors.

II. Discrete-Time Stability of DVFB: Euler's Method

Suppose that the DVFB control law

$$f = -Qy \tag{3}$$

where $Q > 0$, is used in the distributed parameter system (1) under the assumption of collocated force actuators and velocity sensors, i.e., $C = B^T$. This produces a closed-loop system:

$$v_t = A_c v \tag{4}$$

where $A_c = A - BQB^T$. Under certain conditions on A it is known that the energy is dissipated from Eq. (4); hence, if (A,B) is controllable, Eq. (4) is stable in continuous time. However, the digital implementation of Eq. (3) would sample and hold the sensor outputs over the time interval Δt to produce the discrete-time DVFB control law

$$f(k) = -Qy(k) \tag{5}$$

where $f(k)$ is constant over $(k+1)\Delta t \le t \le k\Delta t$. Consequently, the closed-loop system at each time step behaves in the following way:

$$v(k+1) = \Phi_c v(k) \tag{6}$$

where $\Phi_c = U_c(\Delta t)$ and $U_c(\Delta t)$ is the semigroup generated by A_c evaluated on the time step Δt. Although Eq. (4) is guaranteed to be stable, we want to know how large Δt may become without a loss of stability in the actual system.

We will *assume* for simplicity that dim v is *finite*. However, much of the Lyapunov analysis that is carried out here may be extended to infinite-dimensional systems if one is careful about certain concepts; we leave such extensions for future study.

As a first step, we assume that Euler's method of approximation of the time derivative is used:

$$v_t \approx \frac{v(t+\Delta t) - v(t)}{\Delta t} \tag{7}$$

This is often exactly the approximation used in on-line controller implementation. The Euler approximation causes Eq. (6) to become

$$v(k+1) = v(k) + \Delta t A_c v(k) \tag{8}$$

From the continuous-time analysis of DVFB, it is known that A_c is stable. Therefore, given any positive definite Q, there is a positive definite solution P to the Lyapunov equation

$$PA_c + A_c^T P + Q = 0 \tag{9}$$

For discrete-time stability we consider the Lyapunov function

$$V(v(k)) \equiv ½ v^T(k) P v(k)$$

and obtaining

$$\Delta V = -\Delta t [Q - \Delta t A_c^T P A_c] \tag{10}$$

where $\Delta V \equiv V(v(k+1)) - V(v(k))$, we have the following result.

Theorem 1. The discrete-time system Eq. (8) remains stable as long as

$$\Delta t < \frac{\lambda_{\min}(Q)}{\lambda(A^T P A)} \tag{11}$$

Received Jan. 4, 1982.

*Associate Professor. Member AIAA.

where $\lambda_{max}(B)$ is the largest eigenvalue of the positive definite matrix B and similarly for $\lambda_{min}(B)$.

Corollary. If Δt satisfies

$$\Delta t < \frac{\lambda_{min}(Q)}{\lambda_{max}(P)\|A_c\|^2} \tag{12}$$

then Eq. (11) is satisfied and Theorem 1 holds.

It is easy to see that Eq. (12) implies Eq. (11) and may be easier to verify. The proof of Theorem 1 follows from the fact that Eq. (11) guarantees the negative definiteness of the right-hand side of Eq. (10) and use of standard Lyapunov theory.[4] In particular, if Q is chosen as the identity then it will suffice that

$$\Delta t < [\lambda_{max}(A_c^T P A_c)]^{-1}$$

where P satisfies

$$PA_c + A_c^T P + I = 0$$

In general, Q may be chosen to enlarge the range of values of Δt.

Although the result of Theorem 1 is not astounding, it does start to give an indication of the acceptable time-step size for stable digital implementation of DVFB. In the next section, we will extend this idea considerably. Solving the Lyapunov equation (9) can be done by most standard linear systems computer packages.

III. Discrete-Time Stability of Stable Continuous-Time Systems

Suppose that the closed-loop system (4) has been obtained by any acceptable *stabilizing* controller (e.g., DVFB, Modern Modal Control, etc.). Since the system A_c is continuous-time stable, then the Lyapunov equation (9) is satisfied for any positive definite Q. Consider *any* discrete-time implementation, Eq. (6), of the system (4) and determine the acceptable time-step size Δt.

Theorem 2. The discrete-time implementation (6) of the continuous-time stable system (4) remains stable as long as the time-step Δt satisfies

$$0 < \Delta t < T \tag{13}$$

where T is the positive real root of the cubic equation

$$(aT+b)^2 T = c \tag{14}$$

where $a = M_c$, $b = \|A_c\|^2$, and $c = \lambda_{min}(Q)/\lambda_{max}(P)$. Also, $T < (c/b^2)$; i.e., Δt must be smaller than the bound in Eq. (12). The constant M_c is defined in the proof.

The proof of Theorem 2 appears in the Appendix. The cubic (14) can be solved for T once a, b, and c are known.

The above result gives a general condition under which digital implementation of a stably controlled continuous-time system will remain stable. Stability is not the only property which may be lost under digital implementation; other issues are discussed in Refs. 5 and 6.

IV. Conclusions

Stable feedback control of large space structures is essential and most of the proposed control schemes address this issue. However, the stability results are usually given in terms of continuous time even though the actual implementation of the controller will be in discrete time via digital computers. The substance of this Note is that such control algorithms will remain stable when implemented in discrete time under certain restrictions on the size of the time step. Upper bounds on the allowable time-step size are given in Theorems 1 and 2.

Acknowledgments

This research was partially supported by the National Aeronautics and Space Administration under Grant NAG-1-171 and by The Charles Stark Draper Laboratory, Inc., Cambridge, Mass. The author is indebted to Robert Strunce of CSDL for suggesting this research problem.

Appendix: Proof of Theorem 2

From Ref. 7, we obtain

$$\Phi_c = U_c(\Delta t) = I + \Delta t A_c + (\Delta t)^2 L_c(\Delta t)$$

where

$$\|L_c(\Delta t)\| \leq M_c < \infty$$

Take

$$V(v(k)) \equiv \tfrac{1}{2} v^T(k) P v(k)$$

and obtain

$$\Delta V = v^T(k) H v(k)$$

where

$$H = \Phi_c^T P \Phi_c - P = -\Delta t[v^T(k) Q_1 v(k) - (\Delta t)^2 v^T(k) Q_2 v(k)]$$

with

$$Q_1 \equiv Q - \Delta t A_c^T P A_c$$

and

$$Q_2 \equiv Q_c^T P A_c + A_c^T P Q_c + \Delta t Q_c^T P Q_c$$

In order to make $\Delta V < 0$, we must choose Δt so that

$$Q_1 - (\Delta t)^2 Q_2 > 0 \tag{A1}$$

Note that Eq. (A1) is satisfied if

$$\lambda_{min}(Q_1) > (\Delta t)^2 M_c \lambda_{max}(P)(2\|A_c\| + \Delta t M_c) \tag{A2}$$

Furthermore,

$$\lambda_{min}(Q_1) \geq \lambda_{min}(Q) - \Delta t \lambda_{max}(A_c^T P A_c)$$

$$\geq \lambda_{min}(Q) - \Delta t \lambda_{max}(P)\|A_c\|^2 \tag{A3}$$

Therefore Eq. (A1) is satisfied if

$$c > \Delta t \Gamma(\Delta t) \tag{A4}$$

where

$$c = \lambda_{min}(Q)/\lambda_{max}(P)$$

$$\Gamma(\Delta t) = (a\Delta t + b)^2 \qquad a = M_c \qquad b = \|A_c\|$$

Since $\Delta t \Gamma(\Delta t)$ is an increasing function of Δt, in order to satisfy Eq. (A4), we can take any $0 < \Delta t < T$ where T satisfies

$$c/T = (aT+b)^2 \tag{A5}$$

Finally we note that $T < c/b^2$; if not, then $T \geq c/b^2$ and, from Eq. (A5),

$$[(ac/b^2 + b)]^2 \leq (aT+b)^2 = c/T \leq cb^2/c = b^2$$

which is impossible since a and c are positive.

References

[1] Canavin, J., "The Control of Spacecraft Vibrations Using Multivariable Output Feedback," AAS/AIAA Astrodynamics Conference, Palo Alto, Calif., 1978.

[2]Balas, M., "Direct Velocity Feedback Control of LSS," *Journal of Guidance and Control,* Vol. 2, May-June 1979, pp. 252-253.
[3]Aubrun, J-N., "Theory of the Control of Structures by Low-Authority Controllers," *Journal of Guidance and Control,* Vol. 3, Sept.-Oct. 1980, pp. 444-451.
[4]Vidyasagar, M., *Nonlinear Systems Analysis,* Prentice-Hall, Englewood Cliffs, N.J., 1978, Chapt. 5.
[5]Öz, H., Meirovitch, L., and Johnson C.R. Jr., "Some Problems Associated with Digital Control of Dynamic Systems," *Journal of Guidance and Control,* Vol. 3, Nov.-Dec. 1980, pp. 523-528.
[6]Giner, S., "Attitude Control of Large Flexible Spacecraft," M.S. Thesis, MIT, Cambridge, Mass., 1978.
[7]Yosida, K., *Functional Analysis,* Springer, Berlin, 1965, p. 245.

AIAA 82-4258

Single-Axis Spacecraft Attitude Maneuvers Using an Optimal Reaction Wheel Power Criterion

S.B. Skaar* and L.G. Kraige†
Virginia Polytechnic Institute and State University, Blacksburg, Va.

Introduction

THE subject of spacecraft attitude control using motor-driven reaction wheels has received the attention of numerous investigators due to the attractiveness of electricity as opposed to expendable thruster fuel. The topics of special-case, nonoptimal, large-angle maneuvers,[1,2] fine-pointing control,[3,4] and optimal strategies for momentum desaturation[5-7] have been widely studied. But the literature contains few efforts (see Ref. 8 for one example) directed toward the treatment of optimal, large-angle, arbitrary attitude maneuvers. The purpose of this Note is to begin the examination of this problem class by considering the special case of single-axis or slewing maneuvers.

The minimization of the energy consumed by the reaction wheel motors is clearly a worthwhile goal. Although the integral of power over the maneuver time would seem to be the ideal performance index since it represents total mechanical work, it does not yield a unique torque history. Moreover, the total work criterion rewards braking, or negative work, as much as it penalizes positive work. Hence, the index treated herein is the integral of power squared over time.

Minimum Power Formulation for Slewing Maneuvers

Consider the rigid-body configuration of Fig. 1, where $\hat{\ell}$ denotes the principal axis about which the maneuver is to occur, I is the spacecraft mass moment of inertia about $\hat{\ell}$, J is the reaction wheel axial moment of inertial about $\hat{\ell}$, θ and ϕ are wheel and spacecraft inertia angular displacements, and u is the motor torque exerted by the motor on the wheel. The problem to be considered is the determination of $u(t)$ such that the performance index

$$J' = \int_{t_0}^{t_f} P^2 \mathrm{d}t \tag{1}$$

(where P is power) is minimized while boundary conditions on the spacecraft and wheel states at the initial and final times are satisfied. The instantaneous power output of the motor is equal to $u\Omega$, where Ω is the angular velocity of the wheel relative to the spacecraft.

It is convenient to express the integrand of Eq. (1) as a function of the state variables ϕ and θ. To that end, recognize that

$$u = J\ddot{\theta} = -I\ddot{\phi} \tag{2}$$

from which follows

$$u = [IJ/(I+J)](\ddot{\theta}-\ddot{\phi}) \tag{3}$$

The incremental work done by the motor is

$$\mathrm{d}W = u(\mathrm{d}\theta - \mathrm{d}\phi) = u(\mathrm{d}\Delta) \tag{4}$$

where

$$\Delta = \theta - \phi \qquad \dot{\Delta} = \dot{\theta} - \dot{\phi} = \Omega \qquad \ddot{\Delta} = \ddot{\theta} - \ddot{\phi} \tag{5}$$

Substitution of Eq. (3) into Eq. (4) yields

$$\mathrm{d}W = [IJ/(I+J)]\ddot{\Delta}\mathrm{d}\Delta = [IJ/(I+J)]\ddot{\Delta}\dot{\Delta}\mathrm{d}t \tag{6}$$

Now, since $\mathrm{d}W = P\mathrm{d}t$, we are able to write the power P as

$$P = [IJ/(I+J)]\ddot{\Delta}\dot{\Delta} \tag{7}$$

so that Eq. (1) becomes

$$J' = \left[\frac{IJ}{I+J}\right]^2 \int_{t_0}^{t_f} (\ddot{\Delta}\dot{\Delta})^2 \mathrm{d}t \tag{8}$$

Application of the extended Euler-Lagrange equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial F}{\partial \ddot{\Delta}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{\Delta}}\right) + \frac{\partial F}{\partial \Delta} = 0 \tag{9}$$

results in the fourth-order ordinary differential equation

$$\frac{\mathrm{d}^4\Delta}{\mathrm{d}t^4}\left(\frac{\mathrm{d}\Delta}{\mathrm{d}t}\right)^2 + 4\frac{\mathrm{d}^3\Delta}{\mathrm{d}t^3}\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2}\frac{\mathrm{d}\Delta}{\mathrm{d}t} + \left(\frac{\mathrm{d}^2\Delta}{\mathrm{d}t^2}\right)^3 = 0 \tag{10}$$

The lengthy analytical integration of Eq. (10) results in

$$\Delta = K_3\left\{-\cos^3\left[\frac{\cos^{-1}(K_1t-K_2)+4\pi}{3}\right] + \frac{12}{5}\cos^5\left[\frac{\cos^{-1}(K_1t-K_2)+4\pi}{3}\right]\right\} - (K_3/4)[K_1t-K_2] + K_4 \tag{11a}$$

and

$$\dot{\Delta} = K_1K_3\cos^2\left[\frac{\cos^{-1}(K_1t-K_2)+4\pi}{3}\right] - \frac{K_1K_3}{4} \tag{11b}$$

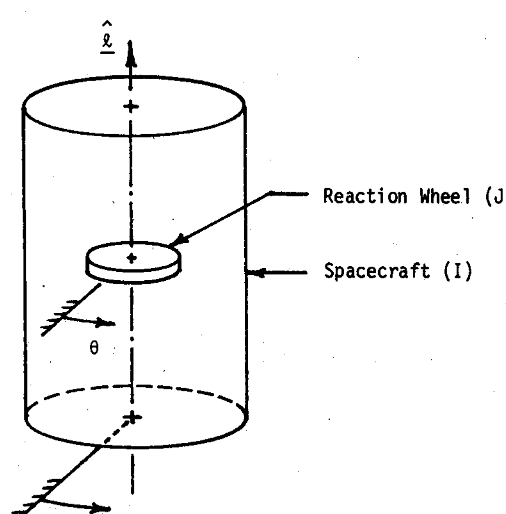

Fig. 1 The spacecraft model.

Received March 2, 1982.
*Instructor, Department of Engineering Science and Mechanics. Associate Member AIAA.
†Associate Professor, Department of Engineering Science and Mechanics. Member AIAA.